General Relativity $\boxed{\text{Week 14}}$

Last time:

__Theorem__ (Choquet-Bruhat, Geroch '69)

Given a smooth initial data set $(\Sigma, \underbrace{\bar{g}, k}_{\text{Satisfy constrained eqns}})$ for the vacuum Einstein equations,

there exists a unique __maximal__ globally hyperbolic development (M, g)

$\hookrightarrow$ i.e if $(\tilde{M}, \tilde{g})$ is any other globally hyperbolic development with corresponding embedding $\tilde{i} : \Sigma \longrightarrow \tilde{M}$, then

$\exists$ isometric embedding $F : \tilde{M} \longrightarrow M$

such that $F \circ \tilde{i} = i$.

Two __important classes of initial data__:

- Asymptotically flat initial data:

  $\exists$ compact subset $K \subset \Sigma$ such that $\Sigma \setminus K$ is a union of domains, each diffeomorphic to $\mathbb{R}^n \setminus B_R$ and, with respect to the usual ~~polar~~ Cartesian coordinates on those

  $$\bar{g}_{ij} = \delta_{ij} + O(|x|^{-1}), \quad k_{ij} = O(|x|^{-2}). \qquad \text{(this can be also weakened).}$$

  __Strongly asymptotically flat__:

  $$\bar{g}_{ij} = \left(1 + \frac{2M}{|x|}\right)\delta_{ij} + o_2(|x|^{-1}), \quad k_{ij} = o_1(|x|^{-2})$$

  where $f = o_k(h)$ means that $|f|, |x||\partial_x f|, \dots, |x|^k|\partial_x^k f| = o(h)$.

They model isolated, self-gravitating systems.

E.g.: • Minkowski: As. flat with 1 end
• Schwarzschild: As. flat with 2 ends (Cart. coordinates at $t=0$: $\bar{g}_{ij} = \left(1+\frac{M}{2|x|}\right)^4 \delta_{ij}$)
• Oppenheimer-Snyder: 1 end

In the case of strongly asymptotically flat initial data:
M plays the role of a "central mass" affecting free-falling observers at $|x| \gg 1$. Can M take negative values?

Positive mass theorem (Schoen-Yau '79, Witten '82)
In the case $n=3$ (also true for $n \leq 8$ or if Σ is spin):
If $(\Sigma^n, \bar{g}, k)$ is strongly asymptotically flat initial data set satisfying the constraint equations for some matter with energy momentum tensor T satisfying the dominant energy condition (i.e if X is fut. directed causal, then $-T^{\alpha}{}_{\beta} X^{\beta}$ is also fut. directed causal)
then at each as. flat end: $M \geq 0$.
If $M=0$ at one end: $(\Sigma, \bar{g}, k)$ is a slice of Minkowski spacetime.

(dominant energy condition at the level of the data:

$$T_{00} \geq \sqrt{\bar{g}^{ij} T_{0i} T_{0j}}$$)

$E \geq |P|$.

Another important result in the class of as. flat initial data:

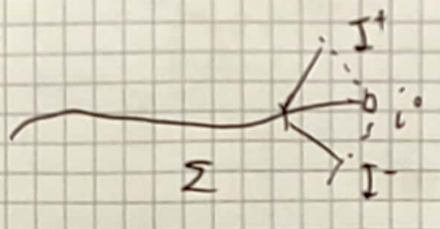
<u>Global stability of Minkowski spacetime</u> (Christodoulou - Klainerman '93)

If $(\Sigma^3, \bar{g}, k)$ is a "small" perturbation of $(\mathbb{R}^3, \delta, 0)$:

The maximal globally hyperbolic development $(M, g) = (\mathbb{R}^{3+1}, g)$ is geodesically complete and, along each timelike geodesic γ, g tends to η.

So: By domain of dependence:

The maximal development of a strongly asymptotically flat initial data set contains a piece of $\mathcal{I}^{\pm}$



- Cosmological initial data:

Σ is compact, $(\bar{g}, k)$ are "almost" homogeneous and isotropic.

(Exactly homogeneous: Einstein equations reduce to ODEs in time).

<u>Incompleteness theorems</u>:

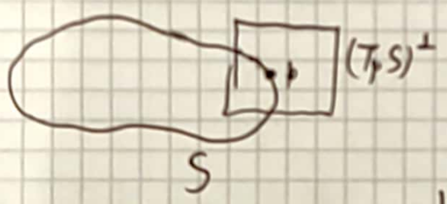
Criteria that guarantee that the maximal globally hyperbolic development (M,g) is not geodesically complete.

<u>Def</u>: (M,g) satisfies the null energy condition if
$$Ric(L,L) \geq 0 \quad \forall \text{ null direction } L$$

(Dominant energy condition $\Longrightarrow$ null energy condition)

Ex: • Vacuum
• T: energy momentum tensor of the scalar field.
• Expected to be true for any realistic matter model.

Let $S^2 \in M^{3+1}$ be a <u>spacelike</u> 2-surface.



Normal bundle $(TS)^\perp$: 2-dimensional and <u>timelike</u> fibers.

Has two null directions:
$$L, \underline{L} \qquad \text{(future directed)}$$

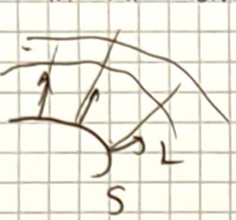
E.g: for the usual round sphere in $(\mathbb{R}^{3+1}, \eta)$:
$$L = \partial_t + \partial_r, \qquad \underline{L} = \partial_t - \partial_r$$

I can define the null second fundamental forms
$$\chi = \langle \nabla_\cdot L, \cdot \rangle, \quad \underline{\chi} = \langle \nabla_\cdot \underline{L}, \cdot \rangle$$

They tell me how the induced metric changes as I "push" S in the direction of L (respectively, $\underline{L}$)

$$\mathcal{L}_L \bar{g}_{ij} = 2\chi_{ij}, \quad \mathcal{L}_{\underline{L}} \bar{g}_{ij} =$$

$\uparrow$ induced metric on S



$\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$: Null expansions

(change of infinitesimal area

$$\mathrm{tr}\chi = \mathcal{L}_L (\log\sqrt{\det \bar{g}}), \quad \mathrm{tr}\underline{\chi} = \ldots$$

For the round sphere in $(\mathbb{R}^{3+1}, \eta)$ of radius R:

$$\mathrm{tr}\chi = \frac{2}{R} > 0,$$

$$\mathrm{tr}\underline{\chi} = -\frac{2}{R} < 0.$$

<u>Def</u>: A <u>compact</u> $S^2 \subset M^{3+1}$ is <u>trapped</u> if

$$\mathrm{tr}\chi < 0 \quad \underline{\text{and}} \quad \mathrm{tr}\underline{\chi} < 0.$$

In Schwarzschild: Every sphere $\{t, r = \text{const}\}$ in the region II is trapped - see Exercise 9.2

<u>Theorem</u>: (Penrose '65)

Let (M^{3+1}, g) be a globally hyperbolic spacetime satisfying the null energy condition. Assume that M has a <u>non-compact</u> Cauchy hypersurface and contains a <u>trapped surface</u> S. Then $\exists$ fut. directed null geodesic starting from S with $\dot{\gamma}(0) \perp S$ which is future incomplete

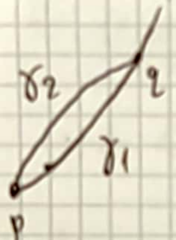
In Schwarzschild: Any such geodesic reaches $r=0$ in finite time

What happens at the "endpoint"?

- Either we reach a singularity
- Or any extension of the spacetime metric is no longer globally hyperbolic (Cauchy horizon - Kerr black hole).

Sketch of the proof:

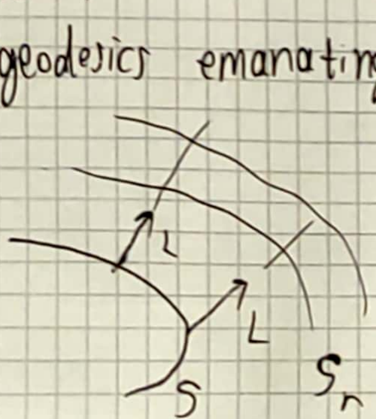
If γ_1, γ_2: future dir. null geodesics emanating from p, reintersect at q:



Any point on γ_1 beyond q: can be connected to p with a <u>timelike</u> curve (so is contained in $I^+(p)$)

For infinitesimally close geodesics: Points of reintersection: Conjugate points to p (i.e. there exists a Jacobi vector field vanishing at p and q).

Let me consider the 3-hypersurface spanned by the null geodesics emanating from S in the direction of L:



$$\underbrace{(r, x^1, x^2)}_{\text{Coordinates on } S} \longrightarrow \exp_{(x^1, x^2)}(rL)$$

S_r: constant r surface ($S_0 = S$)

Induced metric: γ_{ij}

$$\frac{d}{dr}\gamma_{ij} = 2\chi_{ij}$$

$$\Rightarrow \frac{d}{dr}\det\gamma = 2\det\gamma \; \mathrm{tr}_\gamma(\chi)$$

If $\det\gamma \to 0$ at some point: Infinitesimally geodesics intersect.

If $\det\gamma(r_0, x^1, x^2) = 0$: $\forall r > r_0$, $(r, x^1, x^2) \in I^+(S)$.

~~Raychaudhuri~~

<u>Raychaudhuri equation</u>: $Ric_{rr} = -\frac{d}{dr}\mathrm{tr}\chi - |\chi|_\gamma^2$

So null energy condition $\Rightarrow \frac{d}{dr}\mathrm{tr}\chi + |\chi|_\gamma^2 \leq 0$

But $\mathrm{tr}A \leq n \cdot |A|^2$

$$\Rightarrow \frac{d}{dr}\mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 \leq 0$$

Here: We use for the sake of contradiction the assumption that each null geodesic $/\!/ L$ is future complete!

If $\mathrm{tr}\chi|_{r=0} < 0 \Longrightarrow \mathrm{tr}\chi \to \infty$ at some $r_* \leq \frac{2}{|\mathrm{tr}\chi|_{r=0}|}$

$$\mathrm{tr}\chi = \frac{1}{2}\frac{\frac{d}{dr}\det\gamma}{\det\gamma} \Longrightarrow \det\gamma \to 0 \text{ as } r \to r_*.$$

So: $J^+(S) / I^+(S)$ is <u>compact</u>.

Let t be a time function on M with level sets $\simeq \Sigma$.

Project $I^+(S)$ on Σ via the flow of ∇t

$\Rightarrow$ it is compact! but it must also be open! contradiction if Σ is <u>non-compact</u>

Hawking incompleteness theorem:

Let (M^{n+1}, g) be globally hyperbolic with $Ric(X,X) \geq 0$ for any causal vector field X (this is automatically true if the dominant energy condition is satisfied). Assume that, on the initial data $(\Sigma, \bar{g}, k)$, we have

$$\inf_{\Sigma} tr_{\bar{g}} k > 0 \qquad \text{(uniform expansion)}$$

Then every past directed timelike geodesic from Σ is incomplete.

(Relevant in the cosmological setting for a big-bang scenario).

Exam: 21/01, I will send email about the time slots.

We will arrange a Q&A session the week before.

Format: You will be given a few pairs of questions (theory & exercise)

You will be asked to choose one.

Then you will have 20 mins of prep & 20 mins on blackboard.

Materials: Wald Chapters 1-4, 6, 8

Ringstrom: Chapters 8 - 14

From the exercises: Look at the first ones (connections, Killing fields, conserved quantities, energy identities)